Lanczos Method in Synchro Calculation of Eigenpairs and Their Derivatives

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Introduction

IN previous methods of sensitivity analysis, such as the Nelson method[1] and the modal method,[2] the eigenpair derivatives are computed after the eigenpair calculation. The Lanczos method[3] frequently is used to calculate eigenpairs of large n-degree-of-freedom (DOF) systems in which only a small number of the lowest m ($m \gg n$) eigenpairs is required. A typical eigenproblem can be described as

$$(K - \lambda M)\phi = 0, \qquad \phi^T M \phi = 1 \tag{1}$$

where K and M are the stiffness and mass matrices of the structure, respectively, and λ and ϕ are their eigenpair.

We construct a matrix that consists of Lanczos vectors $V = [v_1 \ v_2 \ \cdots \ v_m]$, which leads to

$$V^T M V = T \equiv \begin{bmatrix} \alpha_1 & \beta_1 & & 0 \\ \beta_1 & \ddots & \ddots & \\ & \ddots & \ddots & \beta_{m-1} \\ 0 & & \beta_{m-1} & \alpha_m \end{bmatrix}, \qquad V^T K V = I \tag{2}$$

The original generalized eigenvalue problem becomes a standard eigensolution with a tridiagonal reduced matrix,

$$(I - \lambda T)Y = 0, \qquad Y^T T Y = 1 \tag{3}$$

The elements of T and V can be calculated recursively.[3]

The lower parts of the m eigenvalues and transformed eigenvectors $\phi = VY$ are the lowest parts of eigenpairs of Eq. (1). If we select the transformation of $V^T K V = T$, $V^T M V = I$, we can obtain the highest part of eigenpairs of Eq. (1).

Zhang[2] developed a Lanczos-reduced method to calculate eigenvector derivatives. One of the keys to the Lanczos-reduced method is how to select the initial vector to obtain a small reduced equation. Frequency shift and an iterative formulation were chosen to select the initial vector, and the order of the equation can be reduced to one.

In this Note, the Lanczos method of eigenpair calculation is improved. Lanczos vectors, elements of tridiagonal matrices, and their derivatives are calculated simultaneously. The contracted sensitivity equation is deduced with the contracted eigenequation. A numerical example shows that the synchro-calculation Lanczos method is more computationally efficient than the frequency-shift Lanczos method in the calculation of eigenvector derivatives with respect to one design variable.

Synchro-Calculation Lanczos Method

Taking the derivative of Eq. (1) with respect to p, which can be any design parameter, yields

$$(I - \lambda T)Y' = (\lambda' T + \lambda T')Y \tag{4}$$

$$Y'^T T Y + Y^T T' Y + Y^T T Y' = 0 \tag{5}$$

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Premultiplying Eq. (4) by Y^T [according to Eq. (3), $Y^T(I - \lambda T) = 0$], we obtain $Y^T(\lambda' T + \lambda T')Y = 0$; because $Y^T T Y = 1$, the closed form of eigenvalue derivatives is obtained:

$$\lambda' = -\lambda Y^T T' Y \tag{6}$$

Equations (4) and (5) represent a reduced sensitivity problem, and they can be solved by Nelson's method:

$$Y' = y + \left(-Y^T T y - \tfrac{1}{2} Y^T T' Y\right) Y \tag{7}$$

where y is the particular solution of Eq. (4).

According to $\phi = VY$, we obtain

$$\phi' = V'Y + VY' \tag{8}$$

In the recursion process of the Lanczos method for the calculation of eigenpairs, Lanczos vectors and tridiagonal matrices are simultaneously computed with their derivatives. The algorithm is as follows.

Select the initial vector $\bar{v}_1$, $\bar{v}_1' = 0$, $\hat{K} = K^{-1}$, for $i = 1, 2, \ldots, m$:

$$\bar{v}_i^T K \bar{v}_i = \beta_{i-1}^2, \qquad \frac{\bar{v}_i'^T K \bar{v}_i + \bar{v}_i^T K' \bar{v}_i + \bar{v}_i^T K \bar{v}_i'}{2\beta_{i-1}} = \beta_{i-1}'$$

$$v_i = \frac{\bar{v}_i}{\beta_{i-1}}, \qquad v_i' = \frac{\bar{v}_i' \beta_{i-1} - \bar{v}_i \beta_{i-1}'}{\beta_{i-1}^2}$$

$$\alpha_i = v_i^T M v_i, \qquad \alpha_i' = v_i'^T M v_i + v_i^T M' v_i + v_i^T M v_i'$$

$$\bar{v}_{i+1} = \hat{K} M v_i - \beta_{i-1} v_{i-1} - \alpha_i v_i$$

$$\bar{v}_{i+1}' = \hat{K}(M' v_i + M v_i' - K' \beta_{i-1} v_{i-1} - K' \alpha_i v_i - K' \bar{v}_{i+1}) - \beta_{i-1}' v_{i-1} - \beta_{i-1} v_{i-1}' - \alpha_i' v_i - \alpha_i v_i'$$

$$V' = [v_1' \ v_2' \ \ldots \ v_m'], \qquad T' = V'^T M V + V^T M' V + V^T M V'$$

Clearly, if all of the calculation of ( )′ is removed, the algorithm is the Lanczos method for the eigenproblem.[3]

Numerical Example

To verify the accuracy and efficiency of the synchro Lanczos method, the cantilevered plane frame is used as an example. The finite element model consists of 78 beam elements comprising 74 nodes and 216 DOF, as shown in Fig. 1.

The properties of the beams are listed in Table 1 (Ref. 4). The moment of inertia of the no. 2 element of the frame is selected as a design variable. The order of the Lanczos vector is selected as $m = 8$. The errors of the lowest five eigenvalues are listed in Table 2 [compare with the solution of original eigenequation (1)].

Table 1 Beam properties

Beam	Mass density ρ, kg/m^3	Modulus of elasticity E, N/m^2	Cross-sectional area A, m^2	Moment of inertia I_Z, m^4
Horizontal	2800	7.5E10	0.004	0.0756
Vertical	2800	7.5E10	0.006	0.0756
Diagonal	2800	7.5E10	0.003	0.0756

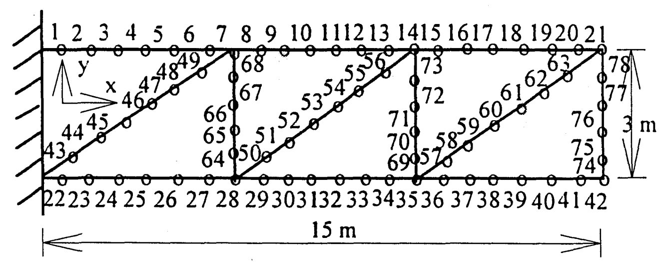

Fig. 1 Plane frame structure: n = 216 DOF (3 DOF/node-72 free nodes) and 1, 2, . . ., element number.

Table 2 Computation error of eigenvalues

Order of eigenvalue	Error, %
First	0.00
Second	0.00
Third	0.00
Fourth	0.00
Fifth	5.01

Table 3 Computation error of eigenvector derivatives

Order of mode	Error, %	
	Synchro	Shift
First	0.49	0.00
Second	0.56	0.00
Third	1.8	4.03
Fourth	2.11	17.58

Table 4 Computational times for eigenpairs and their derivatives

Method	Computational time, s
Synchro Lanczos	98
Shift Lanczos	148
Ratio	66.15/100

The errors of the lowest four eigenvector derivatives are listed in Table 3 (compare with the results calculated by Nelson's method).

When the frequency-shift Lanczos method is used to compute the eigenvector derivatives, select the initial vector so that the order of reduced equation is one, i.e., $T = \alpha_1$. The accuracy of the eigenvector derivatives can be kept at less than 1%. The errors of the third and fourth eigenvector derivatives listed in Table 3 are larger than 1% because of the effect of the error of eigenvector. When a high-precision eigenvector is used to compute eigenvector derivatives, lower accuracy is obtained by the synchro Lanczos method than by the frequency-shift Lanczos method, but the error is still less than 1%. The frequency-shift Lanczos method is more sensitive to the error of eigenvector than the synchro Lanczos method. Computational times are listed in Table 4.

About a one-third savings in computational time over the frequency-shift Lanczos method is observed when using the synchro Lanczos method. The frequency-shift Lanczos method can effectively reduce the order of the system, but it needs to solve a set of system-size equations to select the initial vector for each mode of interest. The synchro Lanczos method can arbitrarily select the initial vector. The result is obtained by selecting an initial vector $\bar{v}_1$ consisting of random data and $\bar{v}_1' = 0$. When selecting $\bar{v}_1$ consisting of the diagonal elements of K and $\bar{v}_1'$ consisting of the diagonal elements of K', there is little difference in the results obtained by the two methods.

Conclusions

The improved Lanczos method can be used to compute the eigenpairs and their derivatives simultaneously. Synchro calculation is the major peculiarity and advantage of this method. It can use not only the final results but also the median results obtained in the preceding eigenanalysis. Higher efficiency is observed when using the present method than when using the frequency-shift Lanczos method. However, as the number of design variables increases, the derivative computation of median matrices increases with each recursion, and the superior efficiency of the present method decreases or even reverses itself. However, synchro calculation provides a new idea for sensitivity analysis.

References

[1]Nelson, R. B., "Simplified Calculation of Eigenvector Derivatives," *AIAA Journal*, Vol. 14, No. 7, 1976, pp. 1201–1205.

[2]Zhang, L. M., He, B. Q., and Yuan, X. R., "Calculation of Mode Shape Derivatives in Structural Dynamic Design: Assessment and Review," *Proceedings of the International Conference on Vibration Engineering* (Beijing, PRC), 1994, pp. 205–210.

[3]Golub, G. H., and Van Loan, C. F., *Matrix Computations*, Johns Hopkins Univ. Press, Baltimore, MD, 1983.

[4]Eckert, L., and Caesar, B., "Model Updating Under Incomplete and Noisy Modal Test Data," *Proceedings of 9th IMAC* (Florence, Italy), 1991, pp. 563–572.

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Virtual Member Method for the Analysis of Frame Structures with Damping Joints

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Introduction

A THREE-DIMENSIONAL frame structure is assembled from one-dimensional members. Vibration of such a structure can be referred to as elastic disturbances propagating in the structure. The elastic waves propagate along members but scatter and reflect at junctions. From this point of view, a traveling-wave model of the structure can be constructed. Because the model is directly obtained from exact boundary conditions at junctions and partial differential equations that govern waveguide motions, the traveling-wave model can provide more accurate dynamic properties of the structure. Furthermore, it is also very convenient for the consideration of the vibration control and the building of a precise local model for a structure.[1–8]

Von Flotow[1] and von Flotow and Schafer[2] presented the traveling-wave model of large flexible spacecraft structures and studied the vibration control problem based on the idea of wave absorbing and wave isolation. Miller and von Flotow[3] and Beale and Accorsi[4] analyzed the power flow in structure networks and gave the energy transmission paths. Vibration control based on a traveling-wave model was also studied by MacMartin et al.[5] and Matsuda and Fujii.[6] Other works also deal with local models, such as complicated junctions[7] and complicated members.[8]

Modern large spacecraft structures (LSS) have been becoming more and more complex and flexible. Dynamic behaviors of LSS need to be accurately controlled to satisfy mission requirements. Because it lacks sufficient damping, such a structure will cause the lessened effectiveness of many envisioned active control methods.[9] For this reason, the damping treatment of a structure is very important. Some of the most popular damping treatment methods are damping layers, damping joints, and damping dashpots.[10]

Damping joints have the advantages of being simple and easy to realize. They can also obtain a good vibration suppression effect.[11] Damping enhancement using active and passive joints has been studied by some researchers.[12,13] In these works, models of the joints are developed mainly in modal space, and only very simple structures, such as beams, have been considered. In practical engineering, the configuration of a joint may be very complicated, and so it is desirable to develop a technique to deal with an arbitrary complex frame structure with complicated joints.

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